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Nonnegative Idempotent Kernels

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The structure of nonnegative idempotent kernels is presented in this paper. The results here generalize (and are based on) earlier results and ideas of Blackwell. Topologized versions of our results along with applications are also given.

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Let X be a state space (possibly uncountable) and $\mathscr{B}$ be a sigma-algebra of subsets of X. Let $Q(x, B)$, $x \in X$ and $B \in \mathscr{B}$, be a nonnegative kernel. This means that for each $x \in X$, $Q(x, \cdot)$ is a (nonnegative) measure on $\mathscr{B}$, and for each $B \in \mathscr{B}$, $Q(\cdot, B)$ is a $\mathscr{B}$-measurable nonnegative function, possibly assuming the value $+\infty$. A set N in $\mathscr{B}$ is called a null set if for every $x \in X$, $Q(x, N) = 0$. A set $I \in \mathscr{B}$ is called invariant if there is a null set N such that $x \in I - N \Rightarrow Q(x, I^c) = 0$. A set $I \in \mathscr{B}$ is called strictly invariant if $Q(x, I^c) = 0$ for each $x \in I$. An invariant set is called indecomposable if it does not contain two disjoint non-null invariant subsets. A nonnegative kernel Q is called idempotent if

$$Q(x, E) = \int Q(y, E)\, Q(x, dy) \tag{2.1}$$

for all $x \in X$ and all $E \in \mathscr{B}$.

Nonnegative idempotent kernels are natural generalizations of infinite dimensional nonnegative idempotent matrices, whose structures are already known [2]. The simplest of such kernels are of the form $Q(x, E) = f(x)\, L(E)$, where f is a strictly positive Borel measurable function, L is a nonnegative Borel measure, and $\int f\, dL = 1$. For example, when X is the set of reals $\geqslant 1$ and m is the Lebesgue measure, then $(1/x^2) \cdot m(E)$ defines such

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a kernel. As we will show in this paper, general nonnegative idempotent kernels are, in most cases, obtained by piecing together kernels of the above type. Such kernels, in a special form, were already studied in the context of groups in [3]. After the authors became aware of Blackwell's work [1], presentation of the theory in [3] became possible in the more general, but natural context that is considered here. In fact, understanding the structure of a general nonnegative idempotent kernel can help us understand with relative ease an apparently nontrivial fact that there cannot be infinite regular idempotent measures in a group. This is made more clear in our Theorem 4.3.

Notice that as in [1],

$$\int_E Q(y, E^c)\, Q(x, dy) = 0 \qquad \text{for all } x \tag{2.2}$$

if and only if for all x, $Q(x, N) = 0$ for the set $N = \{y \in E\colon Q(y, E^c) > 0\}$ if and only if E is invariant. In [1], Blackwell proved the following theorem.

THEOREM 2.1. *Suppose that $\mathscr{B}$ is strictly separable; i.e., it is generated by a denumerable subcollection of its elements. Let P be an idempotent stochastic kernel. Then X can be partitioned in the following manner: $X = N \cup (\bigcup S_\beta)$, where*

(i) *N is a null set and for each β, $S_\beta \in \mathscr{B}$;*

(ii) *for x, y in S_β and $E \in \mathscr{B}$, $P(x, E) = P(y, E)$;*

(iii) *for each β, $P(x, S_\beta) = 1$ for $x \in S_\beta$;*

(iv) *for $x \in N$, $E \in \mathscr{B}$, $E \subset S_\beta$, and $y \in S_\beta$, $P(x, E) = P(y, E)\, P(x, S_\beta)$.*

The structure of a nonnegative idempotent kernel Q, where for each $x \in X$, $Q(x, \cdot)$ is a finite (not necessarily a probability) measure on $\mathscr{B}$, can be easily derived from Blackwell's theorem by a simple trick. (This trick does not seem to work as smoothly when $Q(x, \cdot)$ is not a finite measure, and this infinite case will be treated differently in the next section.) We have the following theorem.

THEOREM 2.2. *Let $\mathscr{B}$ be as in Theorem 2.1. Then X, the state space of a nonnegative idempotent kernel Q, can be partitioned as*

$$X = S \cup N \cup \left(\bigcup S_\beta\right),$$

where

(i) $S = \{x\colon Q(x, X) = 0\}$;

(ii) *N is a null set and for each β, $S_\beta \in \mathscr{B}$;*

(iii) *if* $x, y \in S_\beta$, *then there exists* $u(x, y) > 0$ *such that for any* $E \in \mathscr{B}$, $Q(x, E) = u(x, y)\, Q(y, E)$ *and*

$$\int_{S_\beta} u(z, x)\, Q(x, dz) = 1;$$

(iv) *for each* $x \in S_\beta$, $Q(x, S_\beta^c) = 0$;

(v) *for each* β, *for any* $x \in N$, $y \in S_\beta$, $A \subset S_\beta$, $A \in \mathscr{B}$,

$$Q(x, A) = Q(y, A) \cdot \int_{S_\beta} u(z, y)\, Q(x, dz).$$

Proof. Let S be as stated in the theorem. Since $Q(x, \cdot)$ is a finite measure for each x, the function $f(x) \equiv Q(x, X)$ is a strictly positive real function on $X - S$. For $x \in X - S$, $A \in \mathscr{B}$, and $A \subset X - S$, let us define

$$P(x, A) = \int_A [f(y)/f(x)]\, Q(x, dy). \tag{2.3}$$

Then for any $D \in \mathscr{B}$ and $x \in X$,

$$\begin{aligned}\int_A Q(x, dy)\, I_D(y) &= Q(x, D \cap A)\\ &= \int Q(x, dy)\, Q(y, D \cap A)\\ &= \int Q(x, dy) \left[\int_A I_D(z)\, Q(y, dz) \right].\end{aligned} \tag{2.4}$$

It follows from (2.3) and (2.4) that for $x \in X - S$ and $A \subset X - S$,

$$\begin{aligned}P(x, A) &= \int_A [f(y)/f(x)]\, Q(x, dy)\\ &= \int Q(x, dy) \frac{f(y)}{f(x)} \cdot \left[\int_A \frac{f(z)}{f(y)} \cdot Q(y, dz) \right]\\ &= \int Q(x, dy) \frac{f(y)}{f(x)} \cdot P(y, A) = \int P(y, A)\, P(x, dy).\end{aligned}$$

Also, for $x \in X - S$,

$$P(x, X - S) = [1/Q(x, X)] \int_{X - S} Q(x, dy)\, Q(y, X) = 1.$$

Thus, P is a stochastic kernel on $(X-S, \mathscr{B} \cap (X-S))$. Using Blackwell's theorem, we can now partition $X-S$ as

$$X-S = N \cup \left(\bigcup S_\beta \right),$$

where N is a P-null (and therefore, Q-null) set. Also, for $x, z \in S_\beta$ and $A \in \mathscr{B}$ $(A \subset X-S)$, we have

$$P(x, A) = P(z, A)$$

and therefore,

$$\int_A Q(x, dy) \cdot \frac{f(y)}{f(x)} = \int_A Q(z, dy) \cdot \frac{f(y)}{f(z)}. \tag{2.5}$$

For $B \in \mathscr{B}$, write $g_B(y) = I_B(y)/f(y)$. Let $(g_n^B(y))$ be a sequence of bounded simple functions increasing to $g(y)$ pointwise. By (2.5), we have

$$\int Q(x, dy) \frac{f(y)}{f(x)} \cdot g_n^B(y) = \int Q(z, dy) \frac{f(y)}{f(z)} \cdot g_n^B(y)$$

so that passing to the limit,

$$Q(x, B) = [f(x)/f(z)] \cdot Q(z, B).$$

If $C \in \mathscr{B}$, $C \subset S$, and $x, z \in S_\beta$, then

$$\begin{aligned} Q(x, C) &= \int Q(x, dy)\, Q(y, C) \\ &= \int_{X-S} Q(x, dy)\, Q(y, C) \\ &= \int_{X-S} [f(x)/f(z)] \cdot Q(z, dy)\, Q(y, C) \\ &= [f(x)/f(z)]\, Q(z, C). \end{aligned}$$

The rest of the proof is easy. ▮

If Q is a nonnegative idempotent kernel where $Q(x, \cdot)$ is not necessarily a finite measure, the above method does not carry over easily. Though we will deal with this case in details in the next section (in a different manner), let us indicate one possible approach resulting from this section's method.

Suppose that $B \in \mathscr{B}$, $Q(\cdot, B)$ is real valued and for some x, $Q(x, B) > 0$. Define the set

$$X_B = \{y: Q(y, B) > 0\}.$$

Clearly, for any $y \in X_B$, $Q(y, X_B) > 0$ since

$$Q(y, B) = \int_{X_B} Q(y, dz)\, Q(z, B).$$

If we define for $x \in X_B$ and $A \subset X_B$,

$$P(x, A) = \int_A Q(x, dy) \cdot [Q(y, B)/Q(x, B)],$$

then it can be verified as before that P is a stochastic idempotent kernel on $(X_B, \mathscr{B} \cap X_B)$. Using Blackwell's theorem on P, we can partition X_B as

$$X_B = N_B \cup \left(\bigcup S_\beta \right),$$

where

(i) N_B is a null set with respect to Q on X_B;

(ii) for $x, y \in S_\beta$ and $A \subset X_B$,

$$Q(x, A) = [Q(x, B)/Q(y, B)] \cdot Q(y, A).$$

Using this decomposition and comparing it with similar decompositions for X_C and $X_{B \cup C}$, where C is another set like B, it is possible to treat the "infinite" case. However, we prefer a more basic approach, which we present in the next section.

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As before, let Q be a nonnegative idempotent kernel on a state space X. But here we allow $Q(x, \cdot)$ to be an infinite measure on $\mathscr{B}$. We will show that under a "sigma-finiteness" type condition for Q, Blackwell's arguments can be modified suitably to obtain a structure theorem for Q as before.

Throughout this section, we assume the following condition (*):

There are sets $K_n \subset K_{n+1}$ in $\mathscr{B}$ such that for each x in X, $Q(x, K_n) < \infty$ for all $n \geq 1$, and

$$X - S = \bigcup_{n=1}^{\infty} \{y \in X: Q(y, K_n) > 0\}, \quad \text{where} \quad S = \{x \in X: Q(x, X) = 0\}.$$

Condition $(*)$ is really a uniform sigma-finiteness type condition. Indeed, if

$$M_{nm} = \left\{ y \in X \colon Q(y, K_n) \geqslant \frac{1}{m} \right\},$$

then $X - S = \bigcup \{ M_{nm} = n \geqslant 1, m \geqslant 1 \}$. For any $x \in X$,

$$Q(x, K_n) = \int Q(y, K_n)\, Q(x, dy) \geqslant \frac{1}{m}\, Q(x, M_{nm})$$

so that $Q(x, M_{nm}) < \infty$ for all n and m. Notice that for $A \subset X - S$ and $x \in X - S$,

$$Q(x, A) = \int_{X-S} Q(y, A)\, Q(x, dy).$$

Also, note that for $x \in X - S$, $Q(x, X - S) > 0$. The reason is that for $x \in X - S$, $Q(x, X) > 0$, whereas for $x \in X - S$, $Q(x, X - S) = 0$ implies that $Q(x, S) = \int Q(y, S)\, Q(x, dy) = \int_S Q(y, S)\, Q(x, dy) = 0$ so that $Q(x, X) = 0$, which is impossible. Thus, Q restricted to $X - S$ is non-trivial and idempotent (assuming, of course, $X \neq S$).

It is now clear that we may assume S to be empty. Also, our assumption $(*)$ allows us to assume that there are sets $L_n \subset L_{n+1}$ in $\mathscr{B}$ such that $X = \bigcup_{n=1}^{\infty} L_n$ and for every $x \in X$, $Q(x, L_n) < \infty$.

Now we are ready to get into our results that lead to the structure theorem for Q.

LEMMA 3.1. *Suppose that $Q(x, E) < \infty$. Then for any $A \in \mathscr{B}$,*

$$\int_A \int_{A^c} Q(z, E)\, Q(y, dz)\, Q(x, dy) = \int_{A^c} \int_A Q(z, E)\, Q(y, dz)\, Q(x, dy).$$

Proof. The proof follows from the same arguments used in the proof of Theorem 2 in [1]. Note that the assumption $Q(x, E) < \infty$ makes all the terms in (5) and (6) of [1] finite even in our case. ∎

LEMMA 3.2. *The class $\mathscr{I}$ of invariant sets in $\mathscr{B}$ is a Borel field (sigma-algebra) of subsets of X.*

Proof. It is routine to verify that $\mathscr{I}$ is closed with respect to countable unions. Now let $A \in \mathscr{I}$. Then there exists a null set N such that $Q(y, X - A) = 0$ for any $y \in A - N$. Thus, using the sets K_n (in $(*)$), we have for any $n \geqslant 1$,

$$\int_A \int_{A^c} Q(z, K_n)\, Q(y, dz)\, Q(x, dy) = 0.$$

By Lemma 3.1,

$$\int_{A^c}\int_A Q(z, K_n)\, Q(y, dz)\, Q(x, dy) = 0.$$

Let $K_n' = \{z \in A\colon Q(z, K_n) > 0\}$. Consider the set

$$N(n) = \left\{y\colon \int_{K_n'} Q(z, K_n)\, Q(y, dz) > 0\right\} \cap A^c.$$

Then $N(n)$ is a null set and for $y \in A^c - N(n)$, $Q(y, K_n') = 0$. Let $N = \bigcup_{n=1}^{\infty} N(n)$. Then N is a null set and for $y \in A^c - N$, $Q(y, A) = 0$ because of condition $(*)$. Hence, $A^c \in \mathscr{I}$. ∎

LEMMA 3.3. *Suppose that $Q(x, E) < \infty$ and $Q(x, F) < \infty$ for each $x \in X$. Then the set $A = \{z\colon Q(z, E) < kQ(z, F)\}$ is invariant.*

Proof. By Lemma 3.1,

$$\begin{aligned} k \cdot \int_{A^c}\int_A & Q(z, F)\, Q(y, dz)\, Q(x, dy) \\ &= k \cdot \int_A \int_{A^c} Q(z, F)\, Q(y, dz)\, Q(x, dy) \\ &\leqslant \int_A \int_{A^c} Q(z, E)\, Q(y, dz)\, Q(x, dy) \\ &= \int_{A^c}\int_A Q(z, E)\, Q(y, dz)\, Q(x, dy). \end{aligned}$$

This means that

$$\int_{A^c}\int_A \{kQ(z, F) - Q(z, E)\}\, Q(y, dz)\, Q(x, dy) \leqslant 0.$$

But the above inequality is an equality by the definition of A. Write $N = \{y\colon \int_A [kQ(z, F) - Q(z, E)]\, Q(y, dz) > 0\} \cap A^c$. Then N is a null set. Since $Q(z, E) < kQ(z, F)$ for $z \in A$, $Q(y, A) = 0$ for $y \in A^c - N$. Thus, A^c (and therefore, A by Lemma 3.2) is invariant. ∎

LEMMA 3.4. *Let $E \in \mathscr{B}$, $F \in \mathscr{B}$. Then the set*

$$A = \{z\colon Q(z, E) < k \cdot Q(z, F)\}$$

is invariant.

(This means that the sets $\{z: Q(z, E) \geqslant k \cdot Q(z, F)\}$, $\{z: Q(z, E) \leqslant k \cdot Q(z, F)\}$, and $\{z: Q(z, E) = k \cdot Q(z, F)\}$ are also invariant.)

Proof. Consider the sets L_n considered in the context of condition $(*)$. Define the sets

$$A(n, m) = \left\{z: Q(z, E \cap L_n) < \left(k - \frac{1}{m}\right) Q(z, F \cap L_n)\right\}.$$

Then,

$$A = \bigcup_{r=1}^{\infty} \bigcup_{m > 1/k} \bigcap_{n=r}^{\infty} A(n, m).$$

The lemma follows from Lemmas 3.2 and 3.3. ∎

LEMMA 3.5. *Let A be a strictly invariant and indecomposable set in $\mathscr{B}$. Then for any $E, F \in \mathscr{B}$, and $x, y \in A$,*

$$Q(x, E)\, Q(y, F) = Q(x, F)\, Q(y, E).$$

Proof. Consider the sets L_n considered earlier. Write $E_n = E \cap L_n$ and $F_n = F \cap L_n$. Let $y \in A$. If $Q(y, F) = 0 = Q(y, E)$, then there is nothing to prove. So we assume that at least one of $Q(y, F)$ and $Q(y, E)$ is positive. Suppose that $Q(y, F) > 0$. Then there exists r such that for $n \geqslant r$, $Q(y, F_n) > 0$. Let $k_n \geqslant 0$ such that $k_n Q(y, F_n) = Q(y, E_n)$. Then we have

$$\begin{aligned} 0 &= Q(y, E_n) - k_n Q(y, F_n) \\ &= \int_A [Q(z, E_n) - k_n Q(z, F_n)]\, Q(y, dz) \qquad (\text{since } Q(y, A^c) = 0) \\ &= \int_{A \cap B_n} + \int_{A \cap B_n^c}, \end{aligned} \tag{3.1}$$

where $B_n = \{z: Q(z, E_n) \geqslant k_n \cdot Q(z, F_n)\}$. On $A \cap B_n^c$, $Q(z, E_n) - k_n Q(z, F_n) < 0$; therefore, if $Q(y, A \cap B_n^c) > 0$, then Eq. (3.1) implies that

$$0 < 0$$

since in this case, $Q(y, A \cap B_n)$ must be zero. (Note that $A \cap B_n \in \mathscr{I}$ and $A \cap B_n^c \in \mathscr{I}$; also, A is indecomposable.) Thus,

$$Q(y, A \cap B_n^c) = 0 \qquad \text{and} \qquad Q(y, A \cap B_n) > 0$$

(since $Q(y, A) > 0$). Hence, the set $\{z \in A: Q(z, E_n) = k_n Q(z, F_n)\}$ is non-null (because of Eq. (3.1)). This means that

$$\{z \in A: Q(z, E_n) \neq k_n Q(z, F_n)\}$$

is null, since A is indecomposable. Let $z \in A$. Then,

$$\begin{aligned} Q(z, E_n) &- k_n Q(z, F_n) \\ &= \int_A [Q(x, E_n) - k_n Q(x, F_n)]\, Q(z, dx) = 0. \end{aligned}$$

Passing to the limit as $n \to \infty$, the lemma follows. ∎

THEOREM 3.1. *Let $\mathscr{B}$ be strictly separable. Then X can be partitioned as $X = S \cup N \cup (\bigcup S_\beta)$, where*

(i) $S = \{x: Q(x, X) = 0\}$;

(ii) *N is a null set;*

(iii) *for each β, $S_\beta \in \mathscr{B}$, and for $x \in S_\beta$, $Q(x, S_\beta^c) = 0$;*

(iv) *if $x, y \in S_\beta$, then there exists $u(x, y) > 0$ such that for any $E \in \mathscr{B}$, $Q(x, E) = u(x, y)\, Q(y, E)$.*

Proof. It is no loss of generality to assume that S is empty, since Q restricted to S^c is also idempotent. Suppose that $(F_i)_{i=1}^\infty$ determine $\mathscr{B}$. We may and do assume that the F_i's include the sets L_n (considered earlier) and form a field $\mathscr{F}$. Define the sets

$$I(A, B, k, n) = \left\{x: \frac{k}{n} Q(x, A) \leqslant Q(x, B) < [(k+1)/n]\, Q(x, A)\right\},$$

where $A \in \mathscr{F}$, $B \in \mathscr{F}$, $k = 0, 1, 2, \ldots$, and $n = 1, 2, 3, \ldots$. Let $\mathscr{I}_1$ be the strictly separable Borel field determined by the sets in

$$\mathscr{F}_0 = \{I(A, B, k, n): A, B \in \mathscr{F}, k \geqslant 0, n \geqslant 1\}.$$

By Lemma 3.4, the sets in $\mathscr{I}_1$ are invariant. By Blackwell's Lemma 2, $\mathscr{I}_1$ is atomic; that is, $X = \bigcup X_\beta$, where the X_β's are the atoms of $\mathscr{I}_1$ and each X_β is a countable intersection of elements in $\mathscr{F}_0$ and their complements. Also, $B \in \mathscr{I}_1$ implies either $B \cap X_\beta$ is empty or $X_\beta \subset B$. For each $B \in \mathscr{I}_1$, the field generated by $\mathscr{F}_0$, let $N(B)$ be the null subset of B such that $Q(x, B^c) = 0$ for each x in $B - N(B)$. Then N, the union of the $N(B)$'s, is a null set. For each β, $S_\beta = X_\beta - N$. Since each X_β is an intersection of countably many members of $\mathscr{I}_1$, it is clear that for any $x \in S_\beta$, $Q(x, X - X_\beta) = 0$. Let x and y be in S_β. Since $Q(x, X) > 0$, $Q(y, X) > 0$ and $X = \bigcup_{n=1}^\infty L_n$, there exists some L_m such that

$$0 < Q(x, L_m) < \infty \qquad \text{and} \qquad 0 < Q(y, L_m) < \infty.$$

Let $v \geqslant 0$ and $A \in \mathscr{F}$ such that $Q(x, A) = v \cdot Q(x, L_m)$. Since

$$\{z: Q(z, A) = v \cdot Q(z, L_m)\} \in \mathscr{I}_1$$

and intersects X_β,

$$X_\beta \subset \{z: Q(z, A) = v \cdot Q(z, L_m)\}.$$

This proves that for any $A \in \mathscr{F}$,

$$Q(x, A) = u(x, y) \cdot Q(y, A), \qquad u(x, y) = Q(x, L_m)/Q(y, L_m).$$

It follows that the measures $Q(x, \cdot)$ and $u(x, y)\, Q(y, \cdot)$ coincide on $\mathscr{B}$ since they coincide on the field that generates $\mathscr{B}$. ∎

Here is an example showing that results of this section do not hold without the assumption of a σ-finiteness type of condition. Let $X = [0, \infty)$ and $\mathscr{B}$ be the Borel subsets of X.

Define the nonnegative kernel Q as

$$Q(x, A) = \begin{cases} 0, & \text{if } \; m([0, x] \cap A) = 0; \\ \infty, & \text{otherwise.} \end{cases}$$

Notice that,

$$\{x: Q(x, A) = 0\} = \{x: m([0, x] \cap A) = 0\}$$

is a closed subset of the reals and therefore, belongs to $\mathscr{B}$. Also, Q is idempotent.

(The reason is the following:

Case. $Q(x, A) = 0$. In this case, $Q(y, A) = 0$ if $0 \leqslant y \leqslant x$ and also, $Q(x, (x, \infty)) = 0$. Therefore, $Q(x, A) = \int Q(y, A)\, Q(x, dy)$.

Case. $Q(x, A) = \infty$. In this case, $m([0, x] \cap A) = 2\delta > 0$. If $x - \delta \leqslant y \leqslant x$, then $m([0, y] \cap A) \geqslant \delta \geqslant 0$ so that $Q(y, A) = \infty$. Also, $Q(x, [x - \delta, x]) = \infty$. Hence,

$$Q(x, A) = \infty = \int Q(y, A)\, Q(x, dy).)$$

Now notice that all sets $[0, a]$, $a > 0$ are invariant, since

$$Q(x, (a, \infty)) = 0 \qquad \text{for} \quad 0 \leqslant x \leqslant a.$$

But, $[0, a + b] - [0, a] = (a, a + b]$ is not invariant since

$$Q(x, [0, a]) = \infty \qquad \text{for} \quad a \leqslant x \leqslant a + b.$$

Thus, the invariant sets do not form a σ-algebra anymore.

In the remainder of this section, we discuss the density case, that is, when the nonnegative idempotent kernel Q has the form

$$Q(x, A) = \int_A p_1(x, y)\, m(dy), \tag{3.2}$$

where p_1 is a nonnegative $\mathscr{B} \times \mathscr{B}$ measurable function and m is a finite measure on $\mathscr{B}$. Then exactly as in [1], it can be shown that there is a nonnegative $\mathscr{B} \times \mathscr{B}$ measurable function p such that

$$p(x, y) = \int p(x, z)\, p(z, y)\, m(dz) \tag{3.3}$$

and

$$Q(x, A) = \int_A p(x, y)\, m(dy) \tag{3.4}$$

for $x, y \in X$ and $A \in \mathscr{B}$.

Consider $(X, \mathscr{I}, m)$. Partition

$$X = S \cup X_1 \cup X_2 \cup \cdots \tag{3.5}$$

such that each X_i is an atom in $(X, \mathscr{I}, m)$ and m is non-atomic on $S \in \mathscr{I}$. As in [1], it follows that S is a null set and therefore, we may and do assume that each X_i in (3.5) is strictly invariant. Using our Lemma 3.5, we can then prove easily that

$$p(x_1, y) = a_n(x_1, x_2)\, p(x_2, y) \tag{3.6}$$

for all $y \in X$, $x_1 \in X_n$, $x_2 \in X_n$, and $0 < a_n(x_1, x_2) < \infty$. Let us define the sets

$$U_n = \{ y \in X_n \colon p(x, y) = 0 \}, \qquad V_n = \{ y \notin X_n \colon p(x, y) > 0 \}$$

for $x \in X_n$. Let $U = \bigcup \{ U_n \colon n \geqslant 1 \}$ and $V = \bigcup \{ V_n \colon n \geqslant 1 \}$. Since $Q(x, X_n^c) = 0$ for $x \in X_n$, $m(V_n) = 0$ for $n \geqslant 1$ so that $m(V) = 0$. Also, for $x \in X_n$,

$$Q(x, U_n) = \int_{U_n} p(x, z)\, m(dz) = 0.$$

For $x \in X_m$ $(m \neq n)$, $Q(x, U_n) = 0$ (since X_m is strictly invariant). For $x \in S$,

$$Q(x, U_n) = \int Q(y, U_n)\, Q(x, dy) = \int_{S^c} Q(y, U_n)\, Q(x, dy) = 0.$$

Thus, U_n (for every $n \geqslant 1$) is null. Therefore, U is null. If $m(U_n) > 0$ for some n, then since X_n is an atom, $m(U_n) = m(X_n)$, and therefore, for $x \in X_n$,

$$0 = Q(x, U_n) = \int_{U_n} p(x, y)\, m(dy) = \int_{X_n} p(x, y)\, m(dy)$$
$$= Q(x, X_n) > 0,$$

a contradiction. Thus, $m(U) = 0$.

Now write

$$A_n = X_n - (U \cup V), \qquad F = X - \left(\bigcup_{n=1}^{\infty} X_n \cup V \right) \subset S.$$

We can now state the following:

X can be partitioned as $X = F \cup V \cup A_1 \cup A_2 \cup \cdots$,

where

(i) $p(x, y) = 0$ for $y \in F$;

(ii) F is null and $m(V) = 0$;

(iii) for x, y in A_i, $p(x, y) > 0$, and for x_1, x_2 in A_i, there exists $0 < a_i(x_1, x_2) < \infty$ such that

$$p(x_1, y) = a_i(x_1, x_2)\, p(x_2, y);$$

(iv) each A_i is strictly invariant.

4

In this section, we topologize Blackwell's theorem. This topologizing is not obvious, and yet, it is necessary for certain applications. One such application will be indicated here.

Let X be a completely regular Hausdorff space and $\mathscr{B}$ be the Borel subsets of X generated by its open subsets. Suppose that P is a stochastic idempotent kernel on $X \times \mathscr{B}$ such that

(i) for each $x \in X$, the probability measure $P(x, \cdot)$ can be approximated by its value on compact subsets from inside;

(ii) for each bounded real continuous function f on X,

$$Pf(x) = \int f(y)\, P(x, dy)$$

is a continuous function on X.

From (i) and (ii), it follows that for an open set G,

$$P(x, G)=\sup\left\{\int f_K(y)\, P(x, dy)\right\},$$

where the "sup" is taken over the family

$\{f_K\colon 0 \leqslant f_K \leqslant 1, f_K$ is continuous, $f_K=1$ on K and $f_K=0$ on $X-G$, where K is a compact subset of $G\}$.

Thus, for an open G, $P(x, G)$ is a lower semi-continuous function of x.

LEMMA 4.1. *Let $A \subset X$ be a strictly invariant and indecomposable set. Then, the closed subset $\bar{A}$ is also strictly invariant and indecomposable.*

Proof. By Theorem 5 [1], we have for any $E \in \mathscr{B}$,

$$P(x, E)=P(y, E), \tag{4.1}$$

whenever x, $y \in A$. Let $y \in \bar{A}$. Then there is a net (y_β) in A such that $y_\beta \to y$ and

$$P(y, \bar{A}) \geqslant \limsup_{\beta} P(y_\beta, \bar{A})=1 \qquad (\text{since } y_\beta \in A).$$

(The reason is that $y \to P(y, \bar{A})$ is upper semi-continuous.)

We now claim that for $E \in \mathscr{B}$,

$$P(y, E)=P(x, E), \tag{4.2}$$

whenever $y \in \bar{A}$ and $x \in A$.

To prove this, let $x \in A$, $y \in \bar{A}$, $E \in \mathscr{B}$. Let $v>0$ and (y_β) be a net in A such that $y_\beta \to y$. Using the regularity of the measures $P(x, \cdot)$ and $P(y, \cdot)$, we see that there exist an open set V and a compact set K such that $V \supset E \supset K$ and

$$\begin{aligned} |P(x, V)-P(x, K)| &< \frac{v}{2}, \\ |P(y, V)-P(y, K)| &< \frac{v}{2}. \end{aligned} \tag{4.3}$$

It follows from (4.1) and (4.3) that

$$P(y, E) \geqslant P(y, K) \geqslant \limsup_{\beta} P(y_\beta, K)=P(x, K) \geqslant P(x, E)-\frac{v}{2}$$

and

$$P(y, E) \leqslant P(y, V) \leqslant \liminf_{\beta} P(y_\beta, V) = P(x, V) \leqslant P(x, E) + \frac{v}{2}.$$

Since $v > 0$ is arbitrary, (4.2) follows.

Finally, we use (4.2) to prove that $\bar{A}$ is indecomposable. Suppose that there are disjoint non-null invariant subsets B and C such that $\bar{A} \supset B \cup C$. Then there is $x \in B$ such that $P(x, B) = 1$. By (4.2), $P(y, B) = 1$ for each $y \in \bar{A}$ so that $P(y, C) = 0$ for each $y \in \bar{A}$. This is a contradiction. ∎

LEMMA 4.2. *Let A_1 and A_2 be two disjoint subsets of X, which are both strictly invariant and indecomposable. Then $\bar{A}_1$ and $\bar{A}_2$ are also disjoint.*

Proof. Suppose, if possible, $\bar{A}_1 \cap \bar{A}_2$ is nonempty and contains y. Let $(y_\beta) \subset A_1$ and $(z_q) \subset A_2$ such that $y_\beta \to y$ and $z_q \to y$. Let $x \in A_1$, $z \in A_2$, and $0 < v < \frac{1}{2}$. There are a closed set $B \subset A_1$ and an open set $V \supset A_1$ such that

$$P(x, B) \geqslant P(x, A_1) - v = 1 - v,$$
$$P(z, V) \leqslant P(z, A_1) + v = v.$$

Now we have

$$P(y, A_1) \geqslant P(y, B) \geqslant \limsup P(y_\beta, B) = P(x, B) \geqslant 1 - v;$$

also,

$$P(y, A_1) \leqslant P(y, V) \leqslant \liminf P(z_q, V) = P(z, V) \leqslant v.$$

This is a contradiction. ∎

Let us now state the following topologized version of Blackwell's theorem.

THEOREM 4.1. *Suppose that $\mathscr{B}$ is strictly separable. (This is true, for instance, when the topology is second countable.) Then we can partition the state space X in the following manner:*

$$X = N \cup \left(\bigcup S_\beta\right),$$

where

(i) *N is a null set;*

(ii) *each S_β is closed, strictly invariant and indecomposable and whenever $x, y \in S_\beta$, the probability measures $P(x, \cdot)$ and $P(y, \cdot)$ are equal.*

Let us remark that it follows easily from the proofs of Lemmas 4.1 and 4.2 that the closures of the sets S_β in Theorem 2.1 will be the sets S_β in Theorem 4.1.

We now apply this theorem to prove an important structure theorem for a regular idempotent probability measure m on a locally compact Hausdorff second countable topological semigroup S. A regular probability measure m on the Borel subsets of S is called idempotent if $m = m * m$; i.e.,

$$m(B) = \int m(Bx^{-1})\, m(dx) = \int m(x^{-1}B)\, m(dx), \qquad B \in \mathscr{B},$$

where $Bx^{-1} = \{y \in S\colon yx \in B\}$ and $x^{-1}B = \{y \in S\colon xy \in B\}$.

It is clear that $P(x, B) = m(Bx^{-1})$ and $Q(x, B) = m(x^{-1}B)$ are regular idempotent stochastic kernels. It is easy to verify that S_m, the support of m, is a closed subsemigroup of S and closure $(S_m \cdot S_m) = S_m$. We assume that the translations, i.e., the mappings $x \to xy$ and $x \to yx$, are closed for each $y \in S$. We intend to prove that S_m is a completely simple subsemigroup. (For definition, see [4, pp. 4 and 6].) The completely simple structure of the support of m helps us identify the structure of m. (See [4] for details.)

THEOREM 4.2. *S_m is a completely simple subsemigroup.*

Proof. We use Theorem 4.1 and partition S_m (assuming that $S = S_m$, with no loss of generality) as

$$S_m = N \cup \left(\bigcup S_\beta \right),$$

where N is a null set and each S_β is a closed, strictly invariant and indecomposable set. For x, y in S_β, the measures $P(x, \cdot) = m(\cdot x^{-1})$ and $P(y, \cdot) = m(\cdot y^{-1})$ are the same. Since $\overline{S_m x}$ is the support of $m(\cdot x^{-1})$,

$$\overline{S_m x} = \overline{S_m y} \subset S_\beta, \qquad (x, y \in S_\beta)$$

and therefore, since translations are closed by assumption, $S_m x$ is a minimal (closed) left ideal. Similarly, considering the idempotent stochastic kernel Q, xS_m is a minimal right ideal. By Proposition 2.8 [4], S_m has a kernel (the union of all minimal left ideals) which is completely simple. This kernel is closed and contains $S_m S_m$, which is dense in S_m. The theorem follows. ▮

THEOREM 4.3. *In a locally compact Hausdorff second countable group S, there does not exist an infinite regular measure m such that for Borel sets $B \subset S$,*

$$m(B) = \int m(Bx^{-1})\, m(dx).$$

Proof. Following the same method as above, it is not difficult to see that as before S_m is a closed completely simple subsemigroup of S and hence a group. Since S_m cannot contain any proper left ideal, $S_m = S_\beta$ for some β so that $m(Ex^{-1}) = p(x)\, m(E)$ for $E \subset S_m$ and $x \in S_m$, where $p(x) > 0$ and $\int p(y)\, m(dy) = 1$. The rest of the proof follows as in Lemma 9 of [3].

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